Entanglement distribution maximization over one-side Gaussian noisy channel

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Abstract

The optimization of entanglement evolution for two-mode Gaussian pure states under one-side Gaussian map is studied. Even there isn't complete information about the one-side Gaussian noisy channel, one can still maximize the entanglement distribution by testing the channel with only two specific states.

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Introduction. The study of properties about quantum entanglement has drawn much interest for a long time[1–4]. Although initially quantum information processing(QIP) was studied with discrete quantum states, it was then extended to the continuous variable (CV) quantum states[5]. So far, many concepts and results with 2-level quantum systems have been extended to the continuous variable case with parallel results, such as the quantum teleportation[6], the inseparability criteion[7], the degree of entanglement[8, 9], the entanglement purification[10–12], the entanglement sudden death[13], the characterization of Gaussian maps[14], and so on. However, this does not mean *all* results with 2-level quantum systems can have parallel results for Gaussian states.

Entanglement distribution is the first step towards many novel tasks in quantum communication and QIP[1]. In practice, there is no perfect channel for entanglement distribution. Naturally, how to maximize the entanglement after distribution is an important question in practical QIP. If we distribute the quantum entanglement by sending one part of the entangled state to a remote place through noisy channel, we can use the model of one-side noisy channel, or one-side map.

Given the factorization law presented by Konrad et al[15], such a maximization problem for entanglement distribution over one-side map does not exist for the 2×2 system because any one-side map will produce the same entanglement on the output states provided that the entanglement of the input pure states are same. The result has been experimentally tested[16] and also been extended [17] recently. However, such a factorization does not hold for the continuous variable state as shown below. In this work, we consider the following problem: Initially we have a bipartite Gaussian pure state. Given a one-side Gaussian map (or a one-side Gaussian noisy channel), how to maximize the entanglement of the output state by taking a Gaussian unitary transformation on the input mode before it is sent to the noisy channel. We find that by testing the channel with only two different states, if a certain result is verified, then we can find the right Gaussian unitary transformation which optimizes the entanglement evolution for any input Gaussian pure state. That is to say, we can maximize the output entanglement even though we don't have the full information of the one-side map. In what follows we shall first show by specific example that the factorization law for 2×2 system presented by Konrad et al[15] does not hold for Gaussian states. We then present an upper bound of the entanglement evolution for initial Gaussian pure states. Based on this, we study how to optimize the entanglement evolution over one-side Gaussian

map by taking a local Gaussian unitary transformation to the mode before sent to the noisy channel.

Output entanglement of one-side Gaussian map and single-mode squeezing. Most generally, a two-mode Gaussian pure state is

$$|g(U,V,q)\rangle = U \otimes V|\chi(q)\rangle \tag{1}$$

and $|\chi(q)\rangle = \sqrt{1-q^2}e^{qa_1^\dagger a_2^\dagger}|00\rangle$ $(-1 \leq q \leq 1)$ is a two-mode squeezed state (TMSS). We define map \$ as a Gaussian map which acts on one mode of the state only. A Gaussian map changes a Gaussian state to a Gaussian state only. In whatever reasonable entanglement measure, the entanglement of a Gaussian pure state in the form of Eq.(1) is uniquely determined by q. Therefore, we define the *characteristic* value of entanglement of the Gaussian pure state $\rho(q) = |g(U,V,q)\rangle\langle g(U,V,q)|$ as

$$E[\rho(q)] = |q|^2. \tag{2}$$

On the other hand, any bipartite Gaussian pure state is fully characterized by its covariance matrix (CM). Suppose the CM of state $U \otimes V|\chi(q)\rangle$ is

$$\Lambda = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{3}$$

$|q|^2$ is uniquely determined by $|A|$ (the determinant of the matrix A). So, to compare the entanglement of two Gaussian pure state, we only need to compare $|A|$ value of their covariance matrices.

We start with the projection operator $\hat{T}_k(q_\alpha)$ which acts on mode k only:

$$\hat{T}_k(q_\alpha) = \sum_{n=0}^{\infty} q_\alpha^n|n\rangle\langle n| = q_\alpha^{a_k^\dagger a_k}. \tag{4}$$

This operator has an important mathematical property

$$\hat{T}_k(q_\alpha)(a_k^\dagger, a_k)\hat{T}_k^{-1}(q_\alpha) = (q_\alpha a_k^\dagger, a_k/q_\alpha) \tag{5}$$

which shall be used latter in this paper. For simplicity, we sometimes omit the subscripts of states and/or operators provided that the omission does not affect the clarity.

Define the one-mode squeezed operator $\mathcal{S}(r) = e^{r(a^{\dagger 2} - a^2)}$ where r is a real number and bipartite state $|\psi_r(q_0)\rangle = I \otimes \mathcal{S}(r)|\chi(q_0)\rangle$. We have

Theorem 1. Consider the one-side map $I \otimes \hat{T}(q_1)$ acting on the initial state $|\psi_r(q_0)\rangle$. The entanglement for the outcome state $I \otimes \hat{T}(q_1)|\psi_r(q_0)\rangle$ is a descending function of $|r|$. Mathematically, it is to say that if $|r_1| > |r_2|$ then

$$E[I \otimes \hat{T}(q_1)|\psi_{r_1}(q_0)\rangle] < E[I \otimes \hat{T}(q_1)|\psi_{r_2}(q_0)\rangle]. \tag{6}$$

This theorem actually shows that there isn't a factorization law similar to that in 2×2 states for the continuous variable states, in whatever good entanglement measure. Using Backer-Compbell-Horsdorff (BCH) formula, up to a normalization factor, we have

$$|\psi_r(q_0)\rangle = e^{-\frac{1}{2}{a_1^\dagger}^2 q_0^2 \tanh(2r) + \frac{1}{2}{a_2^\dagger}^2 \tanh(2r) + \frac{q_0 a_1^\dagger a_2^\dagger}{\cosh(2r)}}|00\rangle. \tag{7}$$

Detailed derivation of this identity is given in the appendix. Based on Eq.(4), the one-side map $I \otimes \hat{T}(q_1)$ changes state $|\psi_r(q_0)\rangle$ into

$$|\psi'\rangle = e^{f_1 {a_1^\dagger}^2 + f_2 {a_2^\dagger}^2 + f_3 a_1^\dagger a_2^\dagger}|00\rangle \tag{8}$$

where $f_1 = -\frac{1}{2}q_0^2 \tanh(2r)$, $f_2 = \frac{1}{2}q_1^2 \tanh(2r)$, and $f_3 = \frac{q_0 q_1}{\cosh(2r)}$. Here we have omitted the normalization factor. Since we only need the covariance matrix of state $|\psi'\rangle$, the normalization can be disregarded because it does not change the covariance matrix. The characteristic function of state $\rho' = |\psi'\rangle\langle\psi'|$ has the form

$$C(\alpha_1, \alpha_2) = \mathrm{tr}[\rho' \hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)] = e^{-\frac{1}{2}\bar{\alpha}\Lambda\bar{\alpha}^T} \tag{9}$$

where $\hat{D}_k(\alpha_k) = e^{\alpha_k a_k^\dagger - \alpha_k^* a_k}$ and $\bar{\alpha} = (x_1, y_1, x_2, y_2)$ with $\alpha_k = \frac{1}{\sqrt{2}}(x_k + iy_k)$. Writing Λ here in the form of Eq.(3), we find $A = \mathrm{diag}[b_1, b_2]$, $C = \mathrm{diag}[c_1, c_2]$ and $B = \mathrm{diag}[d_1, d_2]$ with $b_1 = -\frac{1}{2} + \frac{1+2f_2}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, $b_2 = -\frac{1}{2} + \frac{1-2f_2}{1-2f_1-2f_2+4f_1f_2-f_3^2}$, $d_1 = -\frac{1}{2} + \frac{1+2f_1}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, $d_2 = -\frac{1}{2} + \frac{1-2f_1}{1-2f_1-2f_2+4f_1f_2-f_3^2}$, $c_1 = \frac{-f_3}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, $c_2 = \frac{f_3}{1-2f_1-2f_2+4f_1f_2-f_3^2}$. The entanglement in whatever measure of state $|\psi'\rangle$ is a rising functional of $|A|$ and

$$|A| = \frac{1}{4} + \frac{2q_0^2 q_1^2}{1 - 4q_0^2 q_1^2 + q_1^4 + q_0^4(1+q_1^4) + (1-q_0^4)(1-q_1^4)\cosh(4r)}. \tag{10}$$

This is obviously a descending functional of $|r|$.

Upper bound of entanglement evolution. Since $U \otimes I$ and $I \otimes \$$ commute, the unitary operator U places no role in the entanglement evolution under one-side map $I \otimes \$$, and hence we only need consider the initial state $|g(I, V, q)\rangle = I \otimes V|\chi(q)\rangle = |\varphi(q)\rangle$. We also define $\rho^G(q_\alpha) = I \otimes \$(|\varphi(q_\alpha)\rangle\langle\varphi(q_\alpha)|)$.

Using Eq.(5), one easily finds $|\varphi(q = q_a q_b)\rangle = \hat{T}(q_a) \otimes I|\varphi(q_b)\rangle$. Since the operator $\hat{T}(q_a) \otimes I$ and the map $I \otimes \$$ commute, there is:

$$\rho^G(q = q_a q_b) = \hat{T}(q_a) \otimes I \rho^G(q_b) \hat{T}^\dagger(q_a) \otimes I. \tag{11}$$

Using entanglement of formation[9, 18], we can calculate the entanglement of the state of a Gaussian state through its optimal decomposition form[9]. Suppose $\rho^G(q_b)$ has the following optimal decomposition[9]:

$$\rho^G(q_b) = U_1 \otimes U_2 \rho^s(q_0) U_1^\dagger \otimes U_2^\dagger \tag{12}$$

Here U_1, U_2 are two local Gaussian unitaries and ρ^s is in the form

$$\begin{aligned} \rho^s(q_0) = \int & d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2) \\ & \hat{D}(\beta_1, \beta_2)|\chi(q_0)\rangle\langle\chi(q_0)|\hat{D}^\dagger(\beta_1, \beta_2), \end{aligned} \tag{13}$$

where $P(\beta_1, \beta_2)$ is positive definite, $\hat{D}(\beta_1, \beta_2) = \hat{D}_1(\beta_1) \otimes \hat{D}_2(\beta_2)$ is a displacement operator defined as $\hat{D}_k(\beta_k) = e^{\beta_k a_k^\dagger - \beta_k^* a_k}$. According to the definition of optimal decomposition[9, 18], there don't exist any other U_1, U_2 and positive definite functional $P(\beta_1, \beta_2)$ which can decompose $\rho^G(q_b)$ in the form of Eq.(12) with a smaller $|q_0|$. The entanglement of $\rho^G(q_b)$ is equal to that of a TMSS $|\chi(q_0)\rangle$, i.e. q_0^2. For the Gaussian state $\rho^G(q_b)$ with its optimal decomposition of Eq.(12), we define the characteristic value of entanglement of $\rho^G(q_b)$ as $E[\rho^G(q_b)] = |q_0|^2$.

Lemma 1. For any local Gaussian unitary U and operator $\hat{T}(q_a)$, we can find θ, θ' and β'' satisfying

$$\begin{aligned} & \hat{T}(q_a) U_1 \otimes U_2 \cdot \hat{D}(\beta_1, \beta_2)|\chi(q_0)\rangle \\ =& \mathcal{R}(\theta') \otimes \mathcal{R}(\theta) \cdot \hat{D}(\beta_1', \beta_2') \cdot \hat{T}(q_a) \mathcal{S}(r) \otimes U_2 |\chi(q_0)\rangle, \end{aligned} \tag{14}$$

where, $\mathcal{S}(r)$ is a squeezing operator defined earlier, $\mathcal{R}(\theta)$ is a rotation operator defined by $\mathcal{R}(\theta)(a^\dagger, a)\mathcal{R}^\dagger(\theta) = (e^{-i\theta} a^\dagger, e^{i\theta} a)$, β_1', β_2' and β_1, β_2 are related by a certain linear transformation.

Proof: Any local Gaussian unitary operator U_1 can be decomposed into the product form of $\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)$. Also, $\mathcal{S}(r)\mathcal{R}(\theta) \otimes U_2 \cdot \hat{D}(\beta_1, \beta_2) = \hat{D}(\beta_1'', \beta_2'') \cdot \mathcal{S}(r)\mathcal{R}(\theta) \otimes U_2$. Define

$\hat{d} = \hat{T}(q_a) \otimes I \cdot \hat{D}(\beta_1'', \beta_2'') \cdot \hat{T}^{-1}(q_a) \otimes I$, we have

$$\begin{aligned} & \hat{T}(q_a) U \otimes I \cdot \hat{D}(\beta_1, \beta_2)|\chi(q_0)\rangle \\ = & \; \hat{T}(q_a)\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta) \otimes I \cdot \hat{D}(\beta_1, \beta_2)|\chi(q_0)\rangle \\ = & \; \mathcal{R}(\theta') \otimes I \cdot \hat{d} \cdot \hat{T}(q_a)\mathcal{S}(r)\mathcal{R}(\theta) \otimes I|\chi(q_0)\rangle \\ = & \; \mathcal{R}(\theta') \otimes \mathcal{R}(\theta) \cdot \hat{D}(\beta_1', \beta_2') \cdot \hat{T}(q_a)\mathcal{S}(r) \otimes I|\chi(q_0)\rangle. \end{aligned}$$

This completes the proof of Eq.(14). In the second equality above, we have used the fact $\hat{T}(q_a)$ and $\mathcal{R}(\theta')$ commute. Also, $\hat{d}$ there is *not* unitary. However, using BCH formula and the vacuum state property $a_k|00\rangle = 0$, we can always construct a unitary operator $\hat{D}(\beta_1', \beta_2')$ so that the final equality above holds. Here $\beta_1',\ \beta_2'$ are certain linear functions of $\beta_1,\ \beta_2$.

Using Eq.(11) and Eq.(12) with Eq.(14) we have

$$\begin{aligned} & E[\rho^G(q = q_a q_b)] \\ =& E[I \otimes U_2 \cdot \hat{T}(q_a)U_1 \otimes I \rho^s U_1^\dagger \hat{T}^\dagger(q_a) \otimes I \cdot I \otimes U_2^\dagger] \\ =& E\left[\mathcal{R}(\theta_1') \otimes U_2\mathcal{R}(\theta_1)\left(\int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2)\right.\right. \\ & \hat{D}(\beta_1', \beta_2') \cdot \hat{T}(q_a)\mathcal{S}(r_1) \otimes I|\chi(q_0)\rangle\langle\chi(q_0)|\mathcal{S}^\dagger(r_1)\hat{T}^\dagger(q_a) \\ & \left.\left.\otimes I \cdot \hat{D}^\dagger(\beta_1', \beta_2')\right) \mathcal{R}^\dagger(\theta_1') \otimes \mathcal{R}^\dagger(\theta_1)U_2^\dagger\right] \\ \leq& E\left[\int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2)\hat{D}(\beta_1', \beta_2') \cdot \hat{T}(q_a) \otimes I\right. \\ & \left.|\chi(q_0)\rangle\langle\chi(q_0)|\hat{T}^\dagger(q_a) \otimes I \cdot \hat{D}^\dagger(\beta_1', \beta_2')\right] \\ \leq& |q_a q_0|^2 = E[|\chi(q_a)\rangle\langle\chi(q_a)|] \cdot E[\rho^G(q_b)]. \end{aligned} \tag{15}$$

In the third step above we have used theorem 1 for the inequality sign. This gives rise to the second theorem:

Theorem 2. Using the entanglement formation as the entanglement measure, if the entanglement of $\rho^G(q_b)$ is equal to that of TMSS $|\chi(q_0)\rangle$, the entanglement of $\rho^G(q = q_a q_b)$ must be not larger than that of TMSS $|\chi(q_a q_0)\rangle$. Mathematically, it is to say that if $|q| \leq |q_b| \leq 1$ we have

$$\frac{E[I \otimes \$(|\varphi(q)\rangle\langle\varphi(q)|)]}{E[I \otimes \$(|\varphi(q_b)\rangle\langle\varphi(q_b)|)]} \leq \frac{E[|\varphi(q)\rangle\langle\varphi(q)|]}{E[|\varphi(q_b)\rangle\langle\varphi(q_b)|]}. \tag{16}$$

Here $|\varphi(q)\rangle = I \otimes V|\chi(q)\rangle$ as defined earlier, V can be any Gaussian unitary operator. Definitely, the inequality also holds if we replace $|\varphi(q)\rangle$ by $|g(U, V, q)\rangle$ and replace $|\varphi(q_b)\rangle$

by $|g(U', V, q_b)\rangle$, and U, U' can be arbitrary unitary operators. Theorem 2 also gives rise to the following corollary.

Corollary 1. Given the one-side Gaussian map $I \otimes \$$, if the equality sign holds in formula (16) for two specific values q, q_b and $0 < |q| < |q_b| \leq 1$, then the equality sign there holds even q, q_b there are replaced by any q', q'', respectively, as long as $|q'|, |q''| \in [|q|, 1]$.

Proof. For simplicity, we first consider the case where q is replaced by any q'. (1) suppose $|q'| \in [|q|, |q_b|]$. The left side of formula (16) is equivalent to $w' \cdot z'$, and $w' = \frac{E[I\otimes\$(|\varphi(q)\rangle\langle\varphi(q)|)]}{E[I\otimes\$(|\varphi(q')\rangle\langle\varphi(q')|)]}$ and $z' = \frac{E[I\otimes\$(|\varphi(q')\rangle\langle\varphi(q')|)]}{E[I\otimes\$(|\varphi(q_b)\rangle\langle\varphi(q_b)|)]}$. The right side of formula (16) is equivalent to $w \cdot z$ and $w = \frac{E[|\varphi(q)\rangle\langle\varphi(q)|]}{E[|\varphi(q')\rangle\langle\varphi(q')|]}$ and $z = \frac{E[|\varphi(q')\rangle\langle\varphi(q')|]}{E[|\varphi(q_b)\rangle\langle\varphi(q_b)|]}$. Theorem 2 itself says that $w' \leq w$ and $z' \leq z$. If the equality sign holds in formula (16), we have $w' \cdot z' = w \cdot z$ hence we must have $w = w'$ and $z = z'$ which is just corollary 1 in the case q is replaced by q'. (2) Suppose $|q'| > |q_b|$. As we have already known, $\rho^G(q) = \hat{T}(q_a) \otimes I\rho^G(q_b)$. Consider Eq.(14). Unitary U_1 in the optimal decomposition of Eq.(12) must be a rotation operator only, i.e., it contains no squeezing, for, otherwise, according to theorem 1, $E(\rho^G(q'))$ is strictly less than $q_0^2 q_a^2$ which means the equality in formula (16) does not hold.

We denote $q' = q_b/q_c$ and $|q_c| < 1$. We have

$$
\begin{aligned}
& \rho^G(q' = q_b/q_c) \\
= \; & \hat{T}^{-1}(q_c) \otimes I\rho^G(q_b)\left(\hat{T}^{-1}(q_c) \otimes I\right)^\dagger \\
= \; & \hat{T}^{-1}(q_c) \otimes I \cdot \mathcal{R}_1 \otimes U_2 \rho^s \mathcal{R}_1^\dagger \otimes U_2^\dagger \cdot \hat{T}^{-1}(q_c) \otimes I \\
= \; & \mathcal{R}_1 \otimes U_2 \cdot \int d^2\beta_1 d^2\beta_2 P(\beta_1, \beta_2)\hat{D}(\beta_1', \beta_2') \\
& |\chi(q_0/q_c)\rangle\langle\chi(q_0/q_c)|\hat{D}^\dagger(\beta_1', \beta_2') \cdot \mathcal{R}_1^\dagger \otimes U_2^\dagger.
\end{aligned} \tag{17}
$$

Here we have used $\hat{T}^{-1}(q_c) \otimes I|\chi(q_b = q'q_c)\rangle = |\chi(q')\rangle$. We have used the optimal decomposition for $\rho^G(q_b)$ in the second equality, and lemma 1 in the last equality above. Eq.(17) is one possible decomposition of the state $\rho^G(q')$, but not necessarily the optimized decomposition. Therefore, $E[\rho^G(q' = q_b/q_c)] \leq |q_0|^2/|q_c|^2 = |q'|^2/|q_b|^2 \cdot E[\rho^G(q_b)]$. On the other hand, according to theorem 2, we further obtain that $E[\rho^G(q_b = q'q_c)] \leq |q_b|^2/|q'|^2 \cdot E[\rho^G(q')]$. Remark: Since here $|q'| \geq q_b$, sign $\leq$ should be replaced by sign $\geq$ in formula (16), when q is replaced by q'. These two inequalities and result of (1) lead to

$$
\frac{E[\rho^G(q')]}{E[\rho^G(q_b)]} = \frac{E[|\chi(q')\rangle\langle\chi(q')|]}{E[|\chi(q_b)\rangle\langle\chi(q_b)|]}. \tag{18}
$$

for any q' provided that $|q| \leq |q'| \leq 1$. Replacing symbol q' above by symbol q'', we have another equation. Comparing these two equations we conclude corollary 1.

Lemma 2: Given any Gaussian unitaries U, V, we have

$$E[I \otimes \$(U \otimes V|\phi^+\rangle\langle\phi^+|U^\dagger \otimes V^\dagger)] = E[I \otimes \$(|\phi^+\rangle)]. \tag{19}$$

Here $|\phi^+\rangle$ is the maximally entangled state defined as the simultaneous eigenstate of position difference $\hat{x}_1 - \hat{x}_2$ and momentum sum $\hat{p}_1 + \hat{p}_2$, with both eigenvalues being 0. Also, when $q = 1$, the state $|\chi(q)\rangle = |\phi^+\rangle$. We shall use the following fact.

Fact 1: For any local Gaussian unitary operators U and V, we can always find another Gaussian unitary operator $\mathcal{V}$ so that

$$U \otimes V|\phi^+\rangle = \mathcal{V} \otimes I|\phi^+\rangle. \tag{20}$$

Proof: Any local Gaussian unitary operator can be decomposed into the product form of $\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)$. For any TMSS $|\chi(q)\rangle$ we have $\mathcal{R}(\theta_1) \otimes \mathcal{R}(\theta_2)|\chi(q)\rangle = I \otimes \mathcal{R}(\theta_1 + \theta_2)|\chi(q)\rangle$. For the maximally TMSS $|\phi^+\rangle$ we have $\mathcal{S}(r) \otimes \mathcal{S}(r)|\phi^+\rangle = |\phi^+\rangle$, for, the both sides are the simultaneous eigenstates of position difference and momentum sum, with both eigenvalues being 0. This also means $\mathcal{S}(r) \otimes I|\phi^+\rangle = I \otimes \mathcal{S}^\dagger(r)|\phi^+\rangle$. Suppose $V = \mathcal{R}(\theta'_B)\mathcal{S}(r_B)\mathcal{R}(\theta_B)$, then

$$U \otimes V|\phi^+\rangle = \mathcal{V} \otimes I|\phi^+\rangle \tag{21}$$

where $\mathcal{V} = U\mathcal{R}(\theta_B)\mathcal{S}^\dagger(r_B)\mathcal{R}(\theta'_B)$. This completes the proof of Eq.(20). If the equality sign in formula (16) holds, we can apply corollary 1 of theorem 2 through replacing q_b by 1 and we obtain that $E[\rho^G(q')] = |q'|^2 \cdot E[I \otimes \$(|\phi^+\rangle)]$. On the other hand, by using theorem 2 and lemma 2 we have $E[\rho^G(q')] \leq |q'|^2 \cdot E[I \otimes \$(|\phi^+\rangle)]$. This means

$$E[\rho^G(q')] = \max_{\{V'\}}\{E[I \otimes \$(|g(I, V', q')\rangle)]\} \tag{22}$$

where $\rho^G(q') = I \otimes \$(|g(I, V, q')\rangle\langle g(I, V, q')|)$ as defined earlier, $\{V'\}$ is the set containing all single-mode Gaussian unitary transformations. The equality holds for *any* q' provided that the equality of formula(16) holds for two specific values q, q_b and $|q'| \geq |q|$. We arrive at the following major conclusion of this Letter:

Major conclusion: Suppose that we have a TMSS $|\chi(q')\rangle$. We want to maximize the entanglement distribution over a one-side Gaussian map $I \otimes \$$ by taking local Gaussian unitary

operation $I \otimes V'$ before entanglement distribution. Although we don't have complete information of the map $I \otimes \$$, it's still possible for us to find out a specific Gaussian unitary operation V so that the entanglement distribution is maximized over all V', for an initial state $|\chi(q')\rangle$ with *any* $|q'| \geq |q|$, as long as we can find two specific values $|q_b| > |q|$, such that the equality sign in formula (16) holds. Obviously, the conclusion is also correct for any initial state which is a Gaussian pure state.

The conclusion actually says that, in verifying that V can maximize the entanglement distribution for all initial states $\{|\chi(q')\rangle||q'| \geq |q|\}$, we only need to verify the equality sign of formula (16) for two specific values.

Experimental proposal. To experimentally test our major conclusion, we can consider the following beamsplitter channel: Initially, beams 1 and 2 are in a TMSS, which is the initial bipartite Gaussian pure state. Beam 3 is in a squeezed thermal state $\rho_3 = \tilde{S}(u_3)\rho_{th}\tilde{S}^\dagger(u_3)$ here $\tilde{S}(u)$ is a squeezing operator defined by $\tilde{S}(u)(\hat{x},\hat{p})\tilde{S}^\dagger(u) = (u\hat{x}, \hat{p}/u)$ and ρ_{th} is a thermal state whose CM is diag$[b_3, b_3]$. Beam 3 together with the beamsplitter makes the one-side Gaussian channel. A beamsplitter will transform $\hat{x}_2, \hat{x}_3$ by $U_B(\hat{x}_2, \hat{x}_3)U_B^{-1} \longrightarrow (\hat{x}_2, \hat{x}_3)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. In an experiment, we can take, e.g., $q = 0.02$ and $q_b = 0.5$, testing with many different V we should find that the equality sign in formula (16) can hold with $V = \tilde{S}(u_2 = u_3)$. Our major conclusion is verified if we can find that the same $V = \tilde{S}(u_3)$ always maximizes the output entanglement for any input state $|\chi(q')\rangle$ provided that $|q'| \geq 0.02$. Numerical calculation is shown in the following figure.

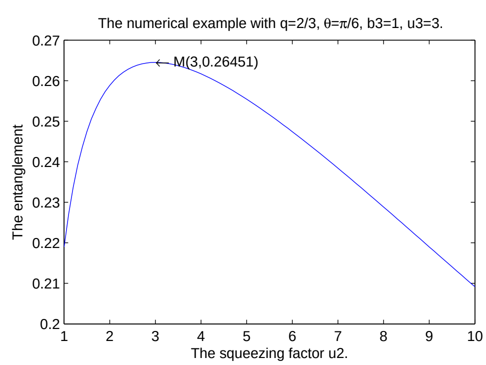

FIG. 1: The entanglement with different squeezing factor u_2. The maximum entanglement obtained when $u_2 = u_3 = 3$. Here we set $u_3 = 3$ and $q' = 2/3, \theta = \pi/6, b_3 = 1$.

In summary, we present an upper bound of the entanglement evolution of a 2-mode Gaussian pure state under one-side Gaussian map. We show that one can maximize the entanglement distribution over an unknown one-side Gaussian noisy channel by testing the channel with only two specific states. An experimental scheme is proposed.

Acknowledgement. This work was supported in part by the National Basic Research Program of China grant nos 2007CB907900 and 2007CB807901, NSFC grant number 60725416, and China Hi-Tech program grant no. 2006AA01Z420.

Appendix. *Details of the proof of Eq.(7).* We will use the following lemma.

Lemma 2. If $\mathcal{A}$ and $\mathcal{B}$ are two noncommuting operators that satisfy the conditions

$$[\mathcal{A},[\mathcal{A},\mathcal{B}]]=[\mathcal{B},[\mathcal{A},\mathcal{B}]]=0, \tag{23}$$

then

$$e^{\mathcal{A}+\mathcal{B}}=e^{\mathcal{A}}e^{\mathcal{B}}e^{-\frac{1}{2}[\mathcal{A},\mathcal{B}]}. \tag{24}$$

This is a special case of the Baker-Hausdorff theorem of group theory[19].

The squeezing operator $S(r)=e^{r(a^{\dagger 2}-a^2)}$ can be normally ordered as[20]

$$\begin{aligned} S(r) &= \frac{1}{\sqrt{\cosh(2r)}}\exp\left[\frac{a^{\dagger 2}}{2}\tanh(2r)\right] \\ &\cdot\exp\left[-a^\dagger a(\ln(\cosh(2r)))\right]\exp\left[-\frac{1}{2}a^2\tanh(2r)\right]. \end{aligned} \tag{25}$$

We neglect the constant of normalization in all the following calculation.

$$\begin{aligned} I\otimes S(r)|\chi(q_0)\rangle &= e^{r(a_2^{\dagger 2}-a_2^2)}e^{q_0a_1^\dagger a_2^\dagger}|00\rangle \\ &= e^{q_0a_1^\dagger(a_2^\dagger\cosh(2r)-a_2\sinh(2r))}e^{r(a_2^{\dagger 2}-a_2^2)}|00\rangle \\ &= e^{q_0a_1^\dagger(a_2^\dagger\cosh(2r)-a_2\sinh(2r))}e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}|00\rangle \\ &= e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{q_0a_1^\dagger\{a_2^\dagger\cosh(2r)-[a_2+a_2^\dagger\tanh(2r)]\sinh(2r)\}}|00\rangle \\ &= e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{q_0a_1^\dagger(\frac{a_2^\dagger}{\cosh(2r)}-a_2\sinh(2r))}|00\rangle \\ &= e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{\frac{q_0a_1^\dagger a_2^\dagger}{\cosh(2r)}}e^{-\frac{1}{2}a_1^{\dagger 2}q_0^2\tanh(2r)}|00\rangle \end{aligned} \tag{26}$$

This is just Eq.(7). In the last equality we have used lemma 2. This completes the proof of

Eq.(7).

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000).

[2] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, Phys. Rev. Lett. 78, 2275 (1997).

[3] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[4] T. Yu, J.H. Eberly, Science 323, 598 (2009).

[5] S. Braunstein and P. van Look, Rev. Mod. Phys. 77, 513 (2005); X.B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, Phys. Rep. 448, 1 (2007).

[6] L. Vaidman, Phys. Rev. A 49, 1473 (1994); S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. 80, 869 (1998); H.F. Hofmann, T. Ide, T. Kobayashi, and A. Furusawa, Phys. Rev. A 62, 062304 (2000); A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science 282,706 (1998).

[7] L.M. Duan, G. Giedke, J.I. Cirac, P. Zoller, Phys. Rev. Lett. 84, 2722 (2000); R. Simon, Phys. Rev. Lett. 84, 2726 (2000); R. F. Werner and M. M. Wolf, Phys. Rev. Lett. 86, 3658 (2001); G. Giedke, B. Kraus, M. Lewenstein, and J.I. Cirac, Phys. Rev. Lett. 87, 167904(2001).

[8] G. Giedke, M. M. Wolf, O. Kruger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. 91, 107901 (2003).

[9] P. Marian and T.A. Marian, Phys. Rev. Lett. 101, 220403 (2008).

[10] Solomon Ivan and R. Simon, arXiv:0808.1658.

[11] J. Eisert, S. Scheel, M.B. Plenio, Phys. Rev. Lett. 89, 137903 (2002).

[12] J. Fiurasek, Phys. Rev. Lett. 89, 137904 (2002).

[13] Juan Pablo Paz and Augusto J. Roncaglia, Phys. Rev. Lett. 100, 220401 (2008); A. S. Coelho AS, F. A. S. Barbosa, K. N. Cassemiro, A. S. Villar, M. Martinelli, and P. Nussenzveig, Science, 326, 823 (2009).

[14] G. Giedke, J.I. Cirac, Phys. Rev. A 66, 032316 (2002).

[15] T Konard, F.D. Melo, M, Tiersch, C. Kansztelan, A. Aragao, and A. Buchleitner, Nature Physics, 4, 99 (2008).

[16] O. Jimenez Farias, C. Lombard Latune, S.P. Walborn, L. Davidovich, P.H. Souto Ribeiro, Science, 324, 1414, (2009).

[17] Chang-shui Yu, X.X. Yi, and He-shan Song, Phys. Rev. A 78, 062330 (2008); Zong-Guo Li, Shao-Ming Fei, Z.D. Wang, and W.M. Liu, Phys. Rev. A 79, 024303 (2009).

[18] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1996).

[19] W.H. Louisell, *Quantum Statistical Properties of Radiation*, (Wiley, New York, 1973).

[20] S.M. Barnett, P.M. Radmore, *Methods in Theoretical Quantum Optics*, (Oxford Science Publication, Oxford, 1997).